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Abstract. Among the many possible ways to study the right tail of a real-valued random variable, a particularly general one is given by considering the family of its Wang distortion risk measures. This class of risk measures encompasses various interesting indicators, such as the widely used Value-at-Risk and Tail Value-at-Risk, which are especially popular in actuarial science, for instance. In this communication, we first build simple extreme analogues of Wang distortion risk measures and we show how this makes it possible to consider many standard measures of extreme risk, including the usual extreme Value-at-Risk or Tail-Value-at-Risk, as well as the recently introduced extreme Conditional Tail Moment, in a unified framework. We then introduce adapted estimators when the random variable of interest has a heavy-tailed distribution and we prove their asymptotic normality. The finite sample performance of our estimators is assessed on a simulation study and we showcase our techniques on an actuarial data set.

Keywords: Asymptotic Normality, Conditional Tail Moment, Distortion Risk Measure, Extreme-Value Statistics, Heavy-Tailed Distribution.

1 Introduction

Understanding the extremes of a random phenomenon is a major question in various areas of statistical application. A stimulating topic comes from the fact that extreme phenomena may have strong adverse effects on financial institutions or insurance companies, and the investigation of those effects on financial returns makes up a large part of the recent extreme value literature.

The most used risk measure is the Value-at-Risk (VaR) which corresponds to the quantile of the loss distribution. Of course, the estimation of a single extreme quantile only gives incomplete information on the extremes of a random variable. To put it differently, it may well be the case that a light-tailed distribution (*e.g.* a Gaussian distribution) and a heavy-tailed distribution share a quantile at some common level, although they clearly do not have the same behavior in their extremes. Besides, the VaR is not a coherent risk measure in the sense of Artzner *et al.* (1999), which is a undesirable feature from the financial point of view. This is why other quantities, which take into account the whole right tail of the random variable of interest, were developed and studied such as the the Conditional Tail Expectation (CTE).

One may then wonder if such risk measures may be encompassed in a single, unified class. An answer, in our opinion, lies in considering Wang distortion risk measures (DRMs), introduced by Wang (1996). The aforementioned VaR and CTE actually are particular cases of Wang DRMs, and so are many other interesting risk measures such as for example the Wang transform (Wang, 2000) which is very popular in finance.

2 Wang risk measures

Let X be a positive random variable. Wang (1996) introduced a family of risk measures called distortion risk measures (DRMs) by the concept of a distortion function: a function $g : [0,1] \to [0,1]$ is a distortion function if it is nondecreasing with $g(0) = 0$ and $g(1) = 1$. The Wang DRM of X with distortion function g is then defined by:

$$R_g(X) := \int_0^\infty g(1 - F(x))dx$$

where F is the cumulative distribution function (cdf) of X. Denote by q the quantile function of X, namely $q(\alpha) = \inf\{x \geq 0 \,|\, F(x) \geq \alpha\}$ for all $\alpha \in (0,1)$. Noticing that $F(x) = \inf\{\alpha \in (0,1) \,|\, q(\alpha) > x\}$ and thus F is the right-continuous inverse of q, a classical change-of-variables formula and an integration by parts then entail that $R_g(X)$, provided it is finite, can be written as a Lebesgue-Stieltjes integral:

$$R_g(X) = \int_0^1 g(\alpha)dq(1-\alpha) = \int_0^1 q(1-\alpha)dg(\alpha).$$

A Wang DRM can thus be understood as a weighted version of the expectation of the random variable X.

3 Extreme versions of Wang risk measures

Extreme versions of Wang risk measures may be obtained as follows. Let g be a distortion function and for every $\beta \in (0,1)$, consider the function $g_\beta(y)$ which is defined by:

$$g_\beta(y) := \begin{cases} g\left(\dfrac{y}{1-\beta}\right) & \text{if} \quad y \le 1-\beta \\ 1 & \text{otherwise} \end{cases} = g\left(\min\left[1, \frac{y}{1-\beta}\right]\right).$$

We can then obtain extreme versions of Wang risk measures as

$$R_{g,\beta}(X) = \int_0^1 q(1-\alpha)dg_\beta(\alpha).$$

The distortion measure dg_β of this risk measure is concentrated on $[0, 1-\beta]$, so that it only takes into account those (high) quantiles of X whose order lies in $[\beta, 1]$. It is actually straightforward to obtain

$$R_{g,\beta}(X) = \int_0^1 q(1-(1-\beta)s)dg(s). \tag{1}$$

3.1 Estimation using an asymptotic equivalent of a Wang DRM

We now give a first idea to estimate this type of extreme risk measure. Let $(X_1, \ldots, X_n)$ be a sample of independent and identically distributed copies of a random variable X having cdf F, and let (β_n) be a nondecreasing sequence of real numbers belonging to $(0,1)$, which converges to 1. Assume X has a heavy-tailed distribution. Recall that a function f is said to be regularly varying at infinity with index $b \in \mathbb{R}$ if f is nonnegative and for any $x > 0$, $f(tx)/f(t) \to x^b$ as $t \to \infty$; the distribution of X is then said to be heavy-tailed when $1-F$ is regularly varying with index $-1/\gamma < 0$, the parameter γ being the so-called tail index of the cdf F. This condition, which is a usual restriction in extremes, essentially says that $1-F(x)$ is in some sense close to $x^{-1/\gamma}$ when x is large. In this case it holds that

$$R_{g,\beta_n}(X) = q(\beta_n)\int_0^1 s^{-\gamma}dg(s)(1+\mathrm{o}(1)) \quad \text{as} \quad n \to \infty$$

provided $\int_0^1 s^{-\gamma-\eta}dg(s) < \infty$ for some $\eta > 0$. This suggests that the above idea for the construction of the estimator can still be used provided n is large enough. Specifically, if $\widehat{F}_n$ denotes the empirical cdf related to this sample and $\widehat{q}_n$ denotes the related empirical quantile function:

$$\widehat{F}_n(x) = \frac{1}{n}\sum_{i=1}^n \mathbb{I}\{X_i \le x\} \quad \text{and} \quad \widehat{q}_n(\alpha) = \inf\{t \in \mathbb{R} \,|\, \widehat{F}_n(t) \ge \alpha\} = X_{\lceil n\alpha \rceil, n}$$

in which $X_{1,n} \le \cdots \le X_{n,n}$ are the order statistics of the sample $(X_1, \ldots, X_n)$ and $\lceil \cdot \rceil$ is the ceiling function, we set

$$\widehat{R}^{AE}_{g,\beta_n}(X) := X_{\lceil n\beta_n \rceil, n}\int_0^1 s^{-\widehat{\gamma}_n}dg(s) \tag{2}$$

where $\widehat{\gamma}_n$ is any consistent estimator of γ (see for example Hill, 1975).

3.2 Estimation using a functional plug-in estimator

The two successive approximations

$$q(1-(1-\beta_n)s) \approx q(\beta_n)s^{-\gamma} \approx X_{\lceil n\beta_n \rceil,n} s^{-\widehat{\gamma}_n},$$

which are at the heart of the construction of the AE estimator, may both introduce substantial errors. Our idea now is to introduce an alternative estimator obtained by making a single approximation, which we can expect to perform better than the AE estimator. We consider the statistic obtained by replacing the function $s \mapsto q(1-(1-\beta_n)s)$ by its empirical counterpart $s \mapsto \widehat{q}_n(1-(1-\beta_n)s) = X_{\lceil n(1-(1-\beta_n)s) \rceil,n}$ in (1). This yields the estimator

$$\widehat{R}^{PL}_{g,\beta_n}(X) = \int_0^1 \widehat{q}_n(1-(1-\beta_n)s)dg(s) \tag{3}$$

which we call the PL estimator. Contrary to the AE estimator, the PL estimator is well-defined and finite with probability 1, and does not require an external estimator of γ. If g is further assumed to be continuous on $[0,1]$, this L-statistic takes the simpler form

$$\widehat{R}^{PL}_{g,\beta_n}(X) = X_{n\beta_n+1,n} + \sum_{i=1}^{n(1-\beta_n)-1} g\left(\frac{i}{n(1-\beta_n)}\right)[X_{n-i+1,n} - X_{n-i,n}].$$

In our first main result we prove the asymptotic normality of our two estimators (AE and PL) which are obtained under the restriction $n(1-\beta_n) \to \infty$. Thus, it only ensures that our estimators consistently estimates so-called intermediate (*i.e.* not "too extreme") Wang DRMs, in the sense that the order of the smallest quantile that it takes into account must converge sufficiently slowly to 1. In other words, our estimators should only be used to estimate those risk measures above a lower threshold $q(\beta_n)$ that belongs to the range covered by the available data. The ultimate goal of our study is to remove this restriction by showing how our extreme-value framework and the expression of our estimators make it possible to use the extrapolation methodology of Weissman (1978) in order to estimate proper extreme Wang DRMs. Our extrapolated estimators are based on the same idea: to estimate the quantity of interest at an arbitrarily extreme level, this quantity is estimated first at an intermediate level where an estimator is known to be consistent, and then multiplied by an extrapolation factor which depends on an external estimate of the tail index γ. Our second main result examines the asymptotic distribution of this type of estimator.

4 Simulation study

The finite-sample performance of our estimators is illustrated on a simulation study, where we consider a couple of classical heavy-tailed distributions (the Fréchet distribution and the Burr distribution) and the following three different distortion functions :

- the Conditional Tail Expectation (CTE) function $g(x) = x$ which weights all quantiles equally;
- the Dual Power (DP) function $g(x) = 1-(1-x)^{1/\alpha}$ with $\alpha \in (0, 1)$, which gives higher weight to large quantiles;
- the Proportional Hazard (PH) transform function $g(x) = x^\alpha$ with $\alpha \in (0, 1)$, which gives higher weight to large quantiles.

5 Real data application

We consider the Secura Belgian Re data set on automobile claims from 1998 until 2001 analyzed in Vandewalle and Beirlant (2006) from the extreme-value perspective. The data set consists of $n = 371$ claims which are at least as large as 1.2 million Euros and were corrected for inflation. Our aim is to revisit this data set and show how we can recover results essentially equivalent to those of Vandewalle and Beirlant (2006) although they worked in a different context.

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